Semiclassical approach to form factors in integrable quantum field theories

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M. L., A. Nesturov, in preparation

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Part I

Basics

Form factors: definition

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Correlation functions are obtained via the spectral decomposition, e.g.

$$\begin{aligned} \langle \mathscr{O}_1(x) \mathscr{O}_2(0) \rangle = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{a_1 \dots a_N} \int \frac{d^N \theta}{(2\pi)^N} \mathrm{e}^{-ix \sum_{i=1}^N p_i} \times \\ \times F_{\mathscr{O}_1}(\theta_1, \dots, \theta_N)_{a_1 \dots a_N} F_{\mathscr{O}_2}(\theta_N - \mathrm{i}\pi, \dots, \theta_1 - \mathrm{i}\pi)_{\bar{a}_N \dots \bar{a}_1}. \end{aligned}$$

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In integrable models sets of form factors can be found exactly by solving a system of equations (M. Karowski, P. Weisz, 1978; F. Smirnov, 1984). But relation between the solutions and particular operators defined in terms of the basic fields is generally problematic.

Action of the sinh-Gordon model (Euclidean):

$$S[\varphi] = \int d^2x \left(\frac{(\partial_\nu \varphi)^2}{16\pi} + \mu \operatorname{ch} b\varphi \right) + \text{const}\,.$$

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$$m = \frac{4\sqrt{\pi}}{\Gamma\left(1+\frac{p}{2}\right)\Gamma\left(\frac{1+p}{2}\right)} \left(-\frac{\pi\Gamma(1+b^2)}{2\Gamma(-b^2)}\mu \right)^{\frac{1}{2+2b^2}} \quad , \quad p = \frac{b^2}{1+b^2} \quad .$$

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Operator contents: exponential operators $\mathrm{e}^{\alpha\varphi}$ and descendants

$$\partial^{k_1}\varphi \cdots \partial^{k_m}\varphi\, \bar\partial^{l_1}\varphi \cdots \bar\partial^{l_n}\varphi\, \mathrm{e}^{\alpha\varphi}, \quad \text{level} = (K, L) = \left(\sum k_i, \sum l_i\right), \text{spin} = K - L.$$

We assume $z = x^1 - x^0 = x^1 + \mathrm{i}x^2$, $\bar z = x^1 + x^0 = x^1 - \mathrm{i}x^2$, $\partial = \partial/\partial z$, $\bar\partial = \partial/\partial \bar z$.

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Reflection equations for exponential operators

$$\mathrm{e}^{\alpha\varphi}=R_\alpha\mathrm{e}^{(Q-\alpha)\varphi}=R_{-\alpha}\mathrm{e}^{-(Q+\alpha)\varphi},\qquad Q=b^{-1}+b$$

with a certain function R_α. For descendant operators the relations are not as explicit.

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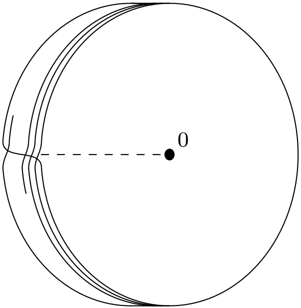

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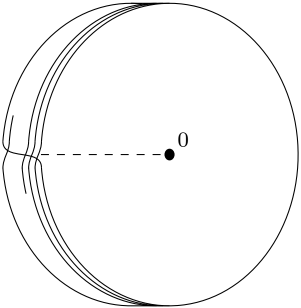

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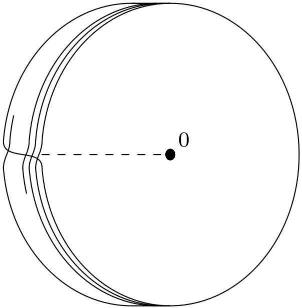

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Twist operator $\mathscr{T}_n(0)$:

$$\mathscr{O}^{(s+1)}(z,\bar{z})\mathscr{T}(0) = \mathscr{O}^{(s)}(z e^{2\pi \mathrm{i}}, \bar{z} e^{-2\pi \mathrm{i}})\mathscr{T}(0)$$

$$\Leftrightarrow \quad \mathscr{O}^{(s)}(x)\mathscr{T}_n(0) = \begin{cases} \mathscr{T}_n(0)\mathscr{O}^{(s)}(x), & x^2 = 0,\ x^1 > 0; \\ \mathscr{T}_n(0)\mathscr{O}^{(s+1)}(x), & x^2 = 0,\ x^1 < 0. \end{cases}$$

Let $\mathscr{O}(\zeta,\bar\zeta)$ be a local operator in the $(\zeta,\bar\zeta)$ plane. Its limit $\zeta,\bar\zeta\to 0$ gives the composite twist operator $\mathscr{T}_n\mathscr{O}(0)$ on the $(z,\bar z)$ plane.

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If $\mathscr{O}$ is a quasiprimary operator of the dimension $(\Delta,\bar\Delta)$ in the UV conformal field theory

$$\mathscr{O}(\zeta,\bar\zeta)=n^{\Delta+\bar\Delta}z^{(1-\frac{1}{n})\Delta}\bar z^{(1-\frac{1}{n})\bar\Delta}\mathscr{O}(z,\bar z).$$

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$$\mathscr{T}_n(\left\{\begin{matrix}\partial^k\varphi\\ \bar\partial^k\varphi\end{matrix}\right\}\mathscr{O})=\left[\mathscr{T}_n(\left\{\frac{\overline{(\partial^k\varphi)_{r_0}}}{(\bar\partial^k\varphi)_{r_0}}\right\}\mathscr{O})(0)+(\text{counterterms})\right]_{r_0\to 0}.$$

Counterterms appear if the limit diverges without them and have the form known from the conformal perturbation theory (Al. Zamolodchikov, 1987).

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Notation: In what follows we omit the superscript (s) assuming

$$x\sim(z,\bar z,s)\sim(\zeta,\bar\zeta).$$

In form factors

$$\theta=(\text{in-particle rapidity})-2\pi\mathrm{i}s.$$

Correlation functions and steepest descent method

In the limit $b \to 0$ the action can be written as

$$S_b[\varphi] = \frac{1}{8\pi} \int d^2x \left(\frac{(\partial_\mu \varphi)^2}{2} + \frac{m^2}{b^2} (\operatorname{ch} b\varphi - 1) \right).$$

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Let $V_\nu(x) = \mathrm{e}^{b^{-1} n\nu\varphi(x)} / \langle \mathscr{T}_n \mathrm{e}^{b^{-1} n\nu\varphi} \rangle$. Consider correlation functions of the form

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Correlation functions and steepest descent method

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Here we may apply the steepest descent method. The saddle point is the solution to the equation

$$\frac{\delta(S_1[\phi]-n\nu\phi(0))}{\delta\phi(x)} = 0 \quad \Leftrightarrow \quad 4\partial_\zeta\partial_{\bar\zeta}\phi - m^2|\zeta|^{2n-2}\operatorname{sh}\phi = -8\pi n\nu\delta(\zeta,\bar\zeta).$$

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Let $z = \zeta^n = r\mathrm{e}^{\mathrm{i}\xi}$, $\bar z = \bar\zeta^n = r\mathrm{e}^{-\mathrm{i}\xi}$. We need a radial solution

$$\phi(x) = \phi_\nu(mr) = -4n\nu\log(m^{1/n}|\zeta|) + O(1) = -4\nu\log mr + O(1), \quad r\to 0.$$

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The classical sinh-Gordon equation

$$4\,\partial\bar{\partial}\phi = \operatorname{sh}\phi$$

can be rewritten in the polar coordinates $mz = t\mathrm{e}^{\mathrm{i}\xi}$, $m\bar{z} = t\mathrm{e}^{-\mathrm{i}\xi}$ in the form

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There is a family of ξ-independent solutions (B. M. McCoy, C. A. Tracy, T. T. Wu, 1977; S. Cecotti, P. Fendley, K. A. Intriligator, C. Vafa, 1992; C. A. Tracy, H. Widom, 1996)

$$\phi_\nu(t) = \sum_{n=0}^{\infty}\lambda^{2n+1}\phi^{(2n+1)}(t), \quad \lambda = \frac{\sin\pi\nu}{\pi},$$

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The function $\phi_\nu(t)$ possesses a reflection property

$$\phi_\nu(t) = \phi_{1-\nu}(t) = -\phi_{-\nu}(t).$$

This property corresponds is the classical counterpart of the above-mentioned reflection property of the quantum model.

Asymptotics of radial solutions

The large t asymptotics of the solutions ϕ_ν are given by

$$\phi_\nu(t) = 2\lambda \int_{-\infty}^{\infty} d\theta\, \mathrm{e}^{-t\,\mathrm{ch}\,\theta} + O\left(\mathrm{e}^{-2t}\right) = \sqrt{8\pi}\lambda t^{-1/2}\mathrm{e}^{-t}\left(1 + O\left(t^{-1}\right)\right), \quad t \to \infty.$$

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$$\mathrm{e}^{\mp\phi_{\pm\nu}(t)/2} = \beta_\nu t^{2\nu} + \beta_{1-\nu} t^{2-2\nu} + O\left(t^2\right) \quad (0 \le \nu \le 1), \qquad t \to 0,$$

where

$$\beta_\nu = 2^{-6\nu} \frac{\Gamma(\frac{1}{2} - \nu)}{\Gamma(\frac{1}{2} + \nu)}.$$

This series can be continued to all orders (E. L. Basor and C. A. Tracy, 1992; Al. Zamolodchikov, 1994; O. Gamayun, N. Iorgov, O. Lisovyy, 2013).

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so that in the classical limit $b \to 0$ we have

$$V_\nu(x) = \frac{\mathrm{e}^{b^{-1} n\nu\varphi(x)}}{\langle \mathscr{T}_n \mathrm{e}^{b^{-1} n\nu\varphi} \rangle}$$

$$\left\langle \prod_{i=1}^{N} \varphi(x_i) \cdot \mathscr{T}_n V_\nu(0) \right\rangle = b^{-N} \prod_{i=1}^{N} \phi_\nu(m r_i) + O(b^{2-N}).$$

Let us write

$$\varphi(x) = b^{-1}\phi_\nu(mr) + \chi(x).$$

Fluctuations around the saddle point

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$$\Delta S[\chi] = \frac{1}{8\pi}\int_{\mathscr{M}_n} d^2x\left(\frac{(\partial_\mu\chi)^2}{2} + \frac{m^2}{b^2}(\operatorname{ch} b\chi - 1)\operatorname{ch}\phi_\nu(mr) + \frac{m^2}{b^2}(\operatorname{sh} b\chi - b\chi)\operatorname{sh}\phi_\nu(mr)\right)$$

$$= \frac{1}{8\pi}\int_{\mathscr{M}_n} d^2x\left(\frac{(\partial_\mu\chi)^2}{2} + \frac{m^2}{2}\chi^2\operatorname{ch}\phi_\nu(mr)\right.$$

$$\left. + \frac{bm^2}{3!}\chi^3\operatorname{sh}\phi_\nu(mr) + \frac{b^2m^2}{4!}\chi^4\operatorname{ch}\phi_\mu(mr) + \cdots\right),$$

Part II

Radial quantization

Recall the radial quantization in CFT. A free massless bosonic field $\varphi_0(x)$ admits a decomposition:

$$\begin{aligned}\varphi_0(x) &= \mathbf{Q} + 2\mathrm{i}\mathbf{P}\log(\zeta\bar\zeta) + \sum_{k\neq 0}\left(\frac{\mathbf{a}_k}{\mathrm{i}k}\zeta^{-k} + \frac{\bar{\mathbf{a}}_k}{\mathrm{i}k}\bar\zeta^{-k}\right) \\ &= \mathbf{Q} + \frac{4\mathrm{i}}{n}\mathbf{P}\log r + \sum_{k\in\frac{1}{n}\mathbb{Z}\setminus\{0\}}\left(\frac{\mathbf{a}_{nk}}{\mathrm{i}nk}\mathrm{e}^{-\mathrm{i}k\xi} + \frac{\bar{\mathbf{a}}_{nk}}{\mathrm{i}nk}\mathrm{e}^{\mathrm{i}k\xi}\right)r^{-k},\end{aligned}$$

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where $\mathbf{P}, \mathbf{Q}, \mathbf{a}_k, \bar{\mathbf{a}}_k$ are operators with the commutation relations

$$[\mathbf{P},\mathbf{Q}] = -\mathrm{i}, \quad [\mathbf{a}_k,\mathbf{a}_l] = [\bar{\mathbf{a}}_k,\bar{\mathbf{a}}_l] = 2k\delta_{k+l}, \quad [\mathbf{a}_k,\bar{\mathbf{a}}_l] = [\mathbf{a}_k,\mathbf{P}] = \cdots = 0. \qquad (1)$$

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The correlation functions are defined as matrix elements between the vacuums:

$$\mathbf{P}|0\rangle = 0, \quad \mathbf{a}_k|0\rangle = \bar{\mathbf{a}}_k|0\rangle = 0 \quad (k>0),$$
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Generalize it to the field χ. Let

$$\chi(x) = \mathbf{Q}f_0(x) + \frac{4\mathrm{i}}{n}\mathbf{P}f_*(x) + \sum_{k\in\frac{1}{n}\mathbb{Z}\setminus\{0\}}\left(\frac{\mathbf{a}_{nk}}{\mathrm{i}nk}f_k(x) + \frac{\bar{\mathbf{a}}_{nk}}{\mathrm{i}nk}\bar f_k(x)\right)$$

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with the same commutation relations (1). Since at $r\to 0$ the operator χ must be indistinguishable from φ_0 we have

$$f_k(x) = m^k u_k(mr)\mathrm{e}^{-\mathrm{i}k\xi}, \quad f_*(x) = u_*(mr),$$
$$u_k(t) = t^{-k}(1+o(1)), \quad u_*(t) = -\log t + O(1) \quad (t\to 0).$$

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By substituting the expansion here we obtain

$$u_k'' + t^{-1}u_k' - (\operatorname{ch}\phi_\nu(t) + k^2 t^{-2})u_k = 0$$

and u_* satisfies this equation with $k = 0$.

Radial quantization: $\nu = 0$

For $\nu = 0$ this equation reduces to the modified Bessel equation

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$$u_k'' + t^{-1}u_k' - (1 + k^2t^{-2})u_k = 0$$

There is a basis that consists of two function $K_k(t)$, $I_k(t)$ for $k \geq 0$:

$$K_k(t) = \frac{1}{2}\int_{-\infty}^{\infty} d\theta\, \mathrm{e}^{k\theta - t\,\mathrm{ch}\,\theta}, \quad I_k(t) = \frac{1}{2\pi}\int_{0+\mathrm{i}\infty}^{2\pi+\mathrm{i}\infty} d\theta\, \mathrm{e}^{\mathrm{i}k\theta + t\cos\theta}.$$

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Their asymptotics

$$K_k(t) = 2^{|k|-1}\Gamma(|k|)t^{-|k|} + O\left(t^{2-|k|}\log t\right), \qquad I_k(t) = \frac{t^k}{2^k\Gamma(k+1)} + O\left(t^{k+2}\right),$$

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as $t \to \infty$. By comparing the small t asymptotics we identify

$$u_*(t) = K_0(t), \quad u_k(t) = \begin{cases} 2^{1-k}\Gamma^{-1}(k)K_k(t), & k > 0; \\ 2^{-k}\Gamma(1-k)\,I_{-k}(t), & k \leq 0. \end{cases}$$

Moreover, the correlation functions $\langle\infty|\chi(x_N)\ldots\chi(x_1)|0\rangle$ are well-defined.

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$$\langle\infty|\chi(x_2)\chi(x_1)|0\rangle = 4\sum_{k\in\frac{1}{n}\mathbb{Z}} K_k(mr_2)I_k(mr_1)\mathrm{e}^{k(\xi_1-\xi_2)}.$$

The series converges. In particular, for $n = 1$ we have

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in accordance with the known result for a massive boson. All other correlation functions are defined by the Wick theorem.

In the general case define the solutions $K_{\nu,k}(t)$, $I_{\nu,k}(t)$ to the generalized Bessel equation ('sinh-Bessel equation')

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by the following conditions:

1. As $t \to 0$ the functions have the asymptotics

$$K_{\nu,k}(t) = \frac{2^{|k|-1}\Gamma(|k|)}{t^{|k|}}(1+o(1)) \quad (k \neq 0),$$
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2. As $t \to \infty$ the functions have the asymptotics

$$\begin{aligned} K_{\nu,k}(t) &= \sqrt{\frac{\pi}{2}}t^{-1/2}\mathrm{e}^{-t}\left(K^\infty_{\nu,|k|} + O\left(t^{-1}\right)\right), \\ I_{\nu,k}(t) &= \frac{1}{\sqrt{2\pi}}t^{-1/2}\mathrm{e}^{t}\left((K^\infty_{\nu,k})^{-1} + O\left(t^{-1}\right)\right). \end{aligned}$$

   Here $K^\infty_{\nu,k}$ is a numeric coefficient.

Radial quantization: $\nu \neq 0$

The functions $K_{\nu,k}$, $I_{\nu,k}$ ($|\nu| \leq \frac{1}{2}$) can be expanded in powers $t^{2\pm 4\nu}$ and $\log t$:

$$I_{\nu,k}(t) = \frac{t^k}{2^k \Gamma(k+1)} \sum_{s,s'\geq 0} c_{\nu,k}^{(s,s')} t^{(2-4\nu)s+(2+4\nu)s'}, \qquad c_{\nu,k}^{(0,0)} = 1,$$

$$K_{\nu,k}(t) = \frac{\pi}{2\sin \pi k} \left(I_{\nu,-k}(t) - \frac{K_{\nu,k}^{\infty}}{K_{\nu,-k}^{\infty}} I_{\nu,k} \right).$$

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The functions $K_{\nu,k}(t)$ admit the series-integral representation

$$\frac{K_{\nu,k}(t)}{K^{\infty}_{\nu,k}} \equiv U_k(t) = \sum_{n=0}^{\infty} \lambda^{2n} U^{(2n+1)}_k(t), \quad U^{(n)}_k(t) = \int d^n\theta\, \mathrm{e}^{k\theta_1} \prod_{i=1}^{n} \frac{\mathrm{e}^{-t\,\mathrm{ch}\,\theta_i}}{2\,\mathrm{ch}\,\frac{\theta_i - \theta_{i+1}}{2}}$$

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Note that $U_k^{(1)}(t) = K_k(t)$.

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Part III

Form factors via the reduction formula. Exponential operators and chiral descendants

Reduction formula

Let us start from the standard reduction formula for a massive boson:

$$a^{+}_{\rm in}(\theta) = \frac{\rm i}{\sqrt{4\pi}} \lim_{x^0\to-\infty} \int_{-\infty}^{\infty} dx^1\, {\rm e}^{{\rm i}p(\theta)x} \overleftrightarrow{\partial}_0\, \varphi(x),$$

$p^0 = m\,{\rm ch}\,\theta$
$p^1 = m\,{\rm sh}\,\theta$

where

$$f(t) \overleftrightarrow{\partial}_t\, g(t) = \frac{1}{2}(f(t)g'(t) - f'(t)g(t)).$$

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Rewriting it in the polar coordinates on the Euclidean plane and applying to a correlation function

$$G_{\mathscr{O}}(\{x_i\}_N) = \langle \varphi(x_N) \cdots \varphi(x_1) \mathscr{T}_n \mathscr{O} \rangle$$

we obtain the form factor

$$F_{\mathscr{O}}(\{\theta_i\}_N) \\ = (-\sqrt{4\pi Z_\varphi})^{-N} \lim_{r_i \to \infty} \left(\prod_{i=1}^{N} r_i \int_{-\pi \pm {\rm i}\infty}^{\pi \pm {\rm i}\infty} d\xi_i \, {\rm e}^{{\rm i}mr_i \operatorname{sh}(\theta_i + {\rm i}\xi_i)} \overleftrightarrow{\partial}_{r_i} \right) G_{\mathscr{O}}(\{x_i\}_N),$$

where Z_φ is the field renormalization factor. In the sinh-Gordon model $Z_\varphi = 1 + O(b^4)$ and can be ignored in the leading order.

Apply this formula to the simplest case of the exponential operator

$$V_\nu(x) = \frac{\mathrm{e}^{b^{-1}n\nu\varphi}}{\langle \mathscr{T}_n \mathrm{e}^{b^{-1}n\nu\varphi} \rangle}.$$

Form factors of the exponential operators

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$$V_\nu(x) = \frac{e^{b^{-1}n\nu\varphi}}{\langle \mathscr{T}_n e^{b^{-1}n\nu\varphi}\rangle}.$$

Its correlation function is given by

$$G_\nu(\{x_i\}_N) = \left\langle \prod_{i=1}^N \varphi(x_i)\cdot \mathscr{T}_n V_\nu(0)\right\rangle = b^{-N}\prod_{i=1}^N \phi_\nu(x_i) + O(b^{2-N}).$$

It is classical in the leading order in b.

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It is classical in the leading order in b. Substitute it into the reduction formula:

$$F_\nu(\{\theta_i\}_N)$$
$$= (-\sqrt{4\pi}b)^{-N} \lim_{r_i \to \infty} \left(\prod_{i=1}^N r_i \int_{-\pi \pm i\infty}^{\pi \pm i\infty} d\xi_i \, e^{imr_i \operatorname{sh}(\theta_i + i\xi_i)} \overleftrightarrow{\partial}_{r_i} \right) \prod_{i=1}^N \phi_\nu(mr_i)$$

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Use the asymptotic formula $\phi_\nu(t)\simeq\sqrt{8\pi}\lambda t^{-1/2}e^{-t}$:

$$\lambda = \frac{\sin\pi\nu}{\pi}$$

$$\cdots = (-\sqrt{2}b^{-1}\lambda m)^N \lim_{r_i\to\infty}\prod_{i=1}^N r_i\int_{-\pi}^{\pi} d\xi_i \left[e^{it\,\mathrm{sh}(\theta_i+i\xi_i)}\overleftrightarrow{\partial}_t\, t^{-1/2}e^{-t}\right]_{t=mr_i}$$

$$= \left(\sqrt{2}b^{-1}\lambda \lim_{t\to\infty}\pi t\overbrace{(I_0(t)+I_0'(t))}^{\sim t^{-1/2}e^{t}}t^{-1/2}e^{-t}\right)^N = \left(\sqrt{4\pi}b^{-1}\lambda\right)^N.$$

Form factors of the exponential operators

Finally we obtain

$$F_\nu(\{\theta_i\}_N) = \left(\sqrt{4\pi}\frac{\sin\pi\nu}{\pi b}\right)^N + O(b^{2-N})$$

in consistency with the $b \to 0$ limit of the exact result.

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Surely, we have the expected reflection properties

$$F_\nu(\{\theta_i\}_N) = F_{1-\nu}(\{\theta_i\}_N) = (-1)^N F_{-\nu}(\{\theta_i\}_N).$$

Form factors of the operators $\partial^k\varphi\, V_\nu$

Consider the operator $\mathscr{T}_n(\partial^k\varphi\, V_\nu)$. Recall that

$$\mathscr{T}_n(\partial^k\varphi\, V_\nu) = \lim_{r_0\to 0} \mathscr{T}_n(\overline{(\partial^k\varphi)}_{r_0} V_\nu).$$

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In the vicinity of it we may consider the field χ as a free massless boson:

$$\varphi(x) \simeq b^{-1}\phi_\nu(m|x|) + \mathbf{Q} + 2\mathrm{i}\mathbf{P}\log(\zeta\bar\zeta) + \sum_{k\neq 0}\left(\frac{\mathbf{a}_k}{\mathrm{i}k}\zeta^{-k} + \frac{\bar{\mathbf{a}}_k}{\mathrm{i}k}\bar\zeta^{-k}\right).$$

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By taking the ∂^k derivative and averaging over a small circle around the origin we obtain

$$\lim_{r_0\to 0} \overline{(\partial^k\varphi)}_{r_0}|0\rangle = \mathrm{i}\,(k-1)!\,\mathbf{a}_{-k}|0\rangle.$$

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Thus the leading contribution to the correlation function is given by

$$\begin{aligned} G_{\partial^k\varphi\, V_\nu}(\{x_i\}_N) &= \mathrm{i}\,(k-1)!\,\langle\infty|\varphi(x_N)\dots\varphi(x_1)\mathbf{a}_{-k}|0\rangle_\nu \\ &= 2\,(k-1)!\sum_{i=1}^N f_{k/n}(x_i)\prod_{j\neq i} b^{-1}\phi_\nu(mr_j). \end{aligned}$$

In the limit $x_i \to \infty$ we get

$$G_{\partial^k\varphi\, V_\nu}(\{x_i\}_N) = G_\nu(\{x_i\}_N) \times b\left(\frac{m}{2}\right)^{\tilde k} \kappa^{(n)}_{\nu,k} \sum_{i=1}^{N} \mathrm{e}^{-\mathrm{i}\tilde k \xi_i},$$

where

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To apply the reduction formula we will need the identity

$$\int_{-\mathrm{i}\pi\pm\mathrm{i}\infty}^{\pi\pm\mathrm{i}\infty} \frac{d\xi}{2\pi}\, \mathrm{e}^{\mathrm{i}t\,\mathrm{sh}(\theta+\mathrm{i}\xi)} \mathrm{e}^{-\mathrm{i}k\xi} = \mathrm{e}^{k(\theta+\frac{\mathrm{i}\pi}{2})} I_{\pm k}(t) \xrightarrow[t\to\infty]{} \frac{\mathrm{e}^{k(\theta+\frac{\mathrm{i}\pi}{2})}}{\sqrt{2\pi}} t^{-1/2}\mathrm{e}^{t}.$$

In the limit $x_i\to\infty$ we get

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Finally, we obtain

$$F_{\partial^k\varphi\,V_\nu}(\{\theta_i\}_N)=b\left(\frac{\mathrm{i}m}{2}\right)^{\tilde k}\kappa^{(n)}_{\nu,k}\sum_{i=1}^N\mathrm{e}^{\tilde k\theta_i}\times\overbrace{F_\nu(\{\theta_i\}_N)}^{\left(\sqrt{4\pi}\frac{\sin\pi\nu}{\pi b}\right)^N}.$$

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$$\kappa^{(n)}_{\nu,k}=\frac{\Gamma(k)}{\Gamma(\tilde k)}\lambda^{-1}K^\infty_{\nu,\tilde k},\qquad \tilde k=\frac{k}{n}.$$

To apply the reduction formula we will need the identity

$$\int_{-\mathrm{i}\pi\pm\mathrm{i}\infty}^{\pi\pm\mathrm{i}\infty}\frac{d\xi}{2\pi}\,\mathrm{e}^{\mathrm{i}t\,\mathrm{sh}(\theta+\mathrm{i}\xi)}\mathrm{e}^{-\mathrm{i}k\xi}=\mathrm{e}^{k(\theta+\frac{\mathrm{i}\pi}{2})}I_{\pm k}(t)\xrightarrow[t\to\infty]{}\frac{\mathrm{e}^{k(\theta+\frac{\mathrm{i}\pi}{2})}}{\sqrt{2\pi}}t^{-1/2}\mathrm{e}^t.$$

Finally, we obtain

$$F_{\partial^k\varphi\,V_\nu}(\{\theta_i\}_N)=b\left(\frac{\mathrm{i}m}{2}\right)^{\tilde k}\kappa^{(n)}_{\nu,k}\sum_{i=1}^N \mathrm{e}^{\tilde k\theta_i}\times\overbrace{F_\nu(\{\theta_i\}_N)}^{\left(\sqrt{4\pi}\frac{\sin\pi\nu}{\pi b}\right)^N}.$$

For $n=k=1$ we immediately reproduce

$$\partial\mathrm{e}^{\alpha\varphi}=\alpha\,\partial\varphi\,\mathrm{e}^{\alpha\varphi}.$$

Naive rule

$\tilde{k} = \frac{k}{n}$

It is suggestive to think that we could obtain form factors of descendants according to the rules:

$$\partial^k \varphi \to b \left(\frac{\mathrm{i}m}{2}\right)^{\tilde{k}} \kappa_{\nu,k}^{(n)} \mathrm{e}^{\tilde{k}\theta_i}, \quad \bar{\partial}^k \varphi \to b \left(\frac{m}{2\mathrm{i}}\right)^{\tilde{k}} \kappa_{\nu,k}^{(n)} \mathrm{e}^{-\tilde{k}\theta_i},$$

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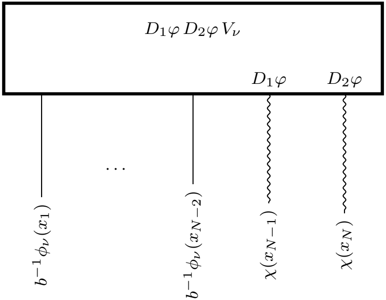
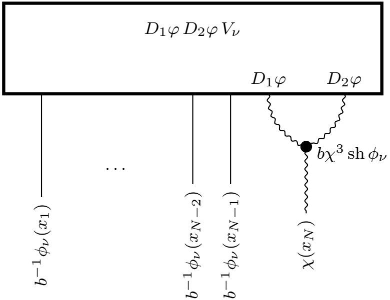
But it contradicts the exact results. What is the problem?
This rule ignores the contributions from the interaction terms, which are of the same order in b as the terms obtained by this rule.

Contribution of the interaction terms

Consider two diagrams:



Order: b^{2-N}



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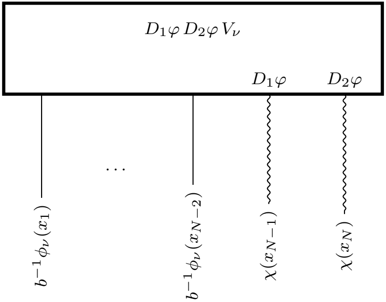

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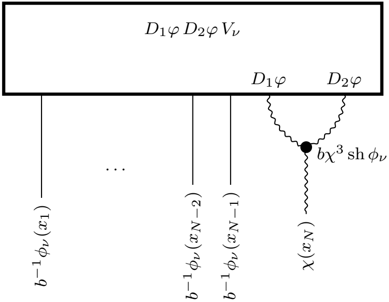

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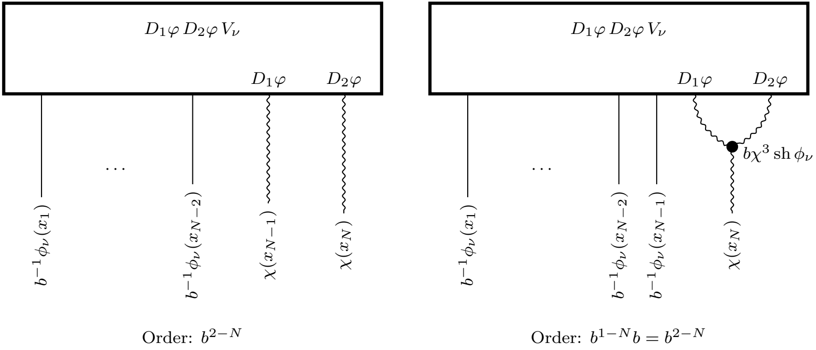

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Explicitly the right diagram looks like

$$-\frac{bm^2}{48\pi}\langle\infty|\,\mathrm{T}_r\left[\chi(x_N)\int d^2y\,{:}\chi^3(y){:}\,\operatorname{sh}\phi_\nu(m|y|)\,\overline{(D_2\chi)}_{r_{02}}\,\overline{(D_1\chi)}_{r_{01}}\right]|0\rangle_\nu\prod_{i=1}^{N-2}b^{-1}\phi_\nu(x_i),$$

where T_r is the r ordering.

In this case we can easily take the limit $r_{01}, r_{02} \to 0$ and obtain

$$
\begin{aligned}
&-\frac{G_{\partial^k\varphi\,\partial^l\varphi\,V_\nu}(\{x_i\}_N)}{(k-1)!\,(l-1)!} \\
&= \langle\infty|\varphi(x_N)\ldots\varphi(x_1)\left(1-\frac{bm^2}{48\pi}\int d^2y\,{:}\chi^3(y){:}\,\mathrm{sh}\,\phi_\nu(m|y|)\right)\mathbf{a}_{-k}\mathbf{a}_{-l}|0\rangle_\nu
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The operators $\partial^k\varphi\,\partial^l\varphi\,V_\nu$

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&= \langle\infty|\varphi(x_N)\ldots\varphi(x_1)\left(1+\mathrm{i}2^{-\delta_{k,l}}\frac{\Gamma(\tilde{k}+\tilde{l})}{\Gamma(\tilde{k}+1)\Gamma(\tilde{l}+1)}b\mathscr{I}^+_{\tilde{k}\tilde{l}}(0)\mathbf{a}_{-k-l}\mathbf{a}_k\mathbf{a}_l\right)\mathbf{a}_{-k}\mathbf{a}_{-l}|0\rangle_\nu \\
&\qquad = \langle\infty|\varphi(x_N)\ldots\varphi(x_1)\left(\mathbf{a}_{-k}\mathbf{a}_{-l}+4\mathrm{i}\frac{\Gamma(\tilde{k}+\tilde{l})}{\Gamma(\tilde{k})\Gamma(\tilde{l})}b\mathscr{I}^+_{\tilde{k}\tilde{l}}(0)\mathbf{a}_{-k-l}\right)|0\rangle_\nu .
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The integrals

$$\mathscr{I}^+_{kl}(t) = \int_t^\infty d\tau\,\tau I_{\nu,k+l}(\tau)K_{\nu,k}(\tau)K_{\nu,l}(\tau)\,\mathrm{sh}\,\phi_\nu(\tau)$$

can be taken analytically and have a finite limit as $t \to 0$ for $k, l \geq 0$.

Integrals $\mathscr{I}^+_{kl}(t)$

The integrals $\mathscr{I}^+_{kl}(t)$ can be expressed in terms of the solution to the equation

$$(\partial_t^2 + t\,\partial_t - \operatorname{ch}\phi_\nu - (k+l)^2 t^{-2}) K_{\nu,kl}(t) = K_{\nu,k}(t) K_{\nu,l}(t) \operatorname{sh}\phi_\nu(t)$$

with the asymptotics

$$K_{\nu,kl}(t) \simeq \frac{K^\infty_{\nu,k} K^\infty_{\nu,l}}{8\lambda} \times \begin{cases} \frac{\Gamma(k+l)}{K^\infty_{\nu,kl}} \left(\frac{2}{t}\right)^{k+l}, & t \to 0; \\ \sqrt{2\pi}\, t^{-1/2} \mathrm{e}^{-t}, & t \to \infty. \end{cases}$$

The connection coefficient reads

$$\frac{1}{K^\infty_{\nu,kl}} = \frac{1}{K^\infty_{\nu,k+l}} + \frac{2^{2k+2l-5}\lambda^2}{\pi\Gamma(k)\Gamma(l)\Gamma(k+l)} \int_{\mathbb{R}^2} dp_1\, dp_2\, \frac{p_1 p_2 \operatorname{sh} 2\pi p_1 \operatorname{sh} \pi p_2}{\prod_{i=1}^2 (\operatorname{ch}^2 \pi p_i - \sin^2 \pi\nu)} \times \prod_{\varepsilon_1,\varepsilon_2=\pm 1} \Gamma\left(\frac{k + \mathrm{i}\varepsilon_1 p_1 + \mathrm{i}\varepsilon_2 p_2}{2}\right) \Gamma\left(\frac{l + \mathrm{i}\varepsilon_1 p_1 + \mathrm{i}\varepsilon_2 p_2}{2}\right).$$

In special cases there are more explicit formulas.

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$$\times\prod_{\varepsilon_1,\varepsilon_2=\pm1}\Gamma\left(\frac{k+\mathrm{i}\varepsilon_1 p_1+\mathrm{i}\varepsilon_2 p_2}{2}\right)\Gamma\left(\frac{l+\mathrm{i}\varepsilon_1 p_1+\mathrm{i}\varepsilon_2 p_2}{2}\right).$$

In special cases there are more explicit formulas.
It turns out that

$$\mathscr{I}^+_{kl}(0) = \frac{K^\infty_{\nu,k}K^\infty_{\nu,l}}{4\lambda}\left(\frac{1}{K^\infty_{\nu,kl}} - \frac{1}{K^\infty_{\nu,k+l}}\right).$$

Form factors of the operators $\partial^k\varphi\,\partial^l\varphi\,V_\nu$

$\tilde k = \frac{k}{n}$

Return to the form factors. We have

$$\frac{F_{\partial^k\varphi\,\partial^l\varphi\,V_\nu}(\{\theta_i\}_N)}{F_\nu(\{\theta_i\}_N)} = b^2\left(\frac{\mathrm{i}m}{2}\right)^{\tilde k+\tilde l}\kappa^{(n)}_{\nu,k}\kappa^{(n)}_{\nu,l}\left(\sum_{i,j=1}^N \mathrm{e}^{\tilde k\theta_i+\tilde l\theta_j} - \frac{K^\infty_{\nu,\tilde k+\tilde l}}{K^\infty_{\nu,\tilde k\tilde l}}\sum_{i=1}^N \mathrm{e}^{(\tilde k+\tilde l)\theta_i}\right).$$

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In particular, for $l=n$ we have $K^\infty_{\nu,\tilde k1}=K^\infty_{\nu,\tilde k}$ and

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For $n=1$ it is consistent with the identity

$$\partial\left(\partial^k\varphi\,\mathrm{e}^{\alpha\varphi}\right) = \partial^{k+1}\varphi\,\mathrm{e}^{\alpha\varphi} + \alpha\,\partial^k\varphi\,\partial\varphi\,\mathrm{e}^{\alpha\varphi}$$

The operators $\partial^k\varphi\,\bar\partial^l\varphi\,V_\nu$ $(k>l)$

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- The circle average of the operator $\bar\partial^l\varphi$ contains an annihilation operator, which contributes the term with $S^{(3)}$:

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- The $S^{(3)}$ contribution contains the integrals

$$\mathscr I^-_{kl}(t)=\int_t^\infty d\tau\,\tau I_{\nu,k-l}(\tau)K_{\nu,k}(\tau)K_{\nu,l}(\tau)\,\mathrm{sh}\,\phi_\nu(\tau),$$
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  which contribute the terms divergent at $r_0\to0$. This demands renormalization.

After the renormalization for generic values of ν,k,l the answer looks simple:

$$\frac{F_{\partial^k\varphi\,\bar\partial^l\varphi\,V_\nu}(\{\theta_i\}_N)}{F_\nu(\{\theta_i\}_N)}=\mathrm{i}^{\tilde k-\tilde l}b^2\left(\frac m2\right)^{\tilde k+\tilde l}\kappa^{(n)}_{\nu,k}\kappa^{(n)}_{\nu,l}\left(\sum_{i,j=1}^N\mathrm e^{\tilde k\theta_i-\tilde l\theta_j}-\frac{K^\infty_{\nu,\tilde k-\tilde l}}{K^\infty_{\nu,\tilde k.-\tilde l}}\sum_{i=1}^N\mathrm e^{(\tilde k-\tilde l)\theta_i}\right).$$

There are special points related to the renormalization thresholds that I omit here.

This case is more special, because even in the term without interaction $\overline{(\bar\partial^k\varphi)_{r_0}}\mathbf{a}_{-k}|0\rangle_\nu = 2\mathrm{i}\overline{(\bar\partial^k f_{\tilde k})_{r_0}}|0\rangle_\nu \neq 0.$
The ratio of the connection coefficients reads

$$\frac{K^\infty_{\nu,0}}{K^\infty_{\nu,k.-k}} = \frac{\cos^2\frac{\pi k}{2}}{\cos\pi\left(\frac{k}{2}+\nu\right)\cos\pi\left(\frac{k}{2}-\nu\right)}.$$

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After the renormalization the form factors read

$$\frac{F^{(n)}_{\partial^k\varphi\,\bar\partial^k\varphi\,V_\nu}(\{\theta_i\}_N)}{F^{(n)}_\nu(\{\theta_i\}_N)} = b^2\left(\frac{m}{2}\right)^{2\tilde k}\varkappa^{(n)\,2}_{\nu,k}$$
$$\times\left(\sum_{i,j=1}^N \mathrm{e}^{\tilde k(\theta_i-\theta_j)} - \frac{K^\infty_{\nu,0}}{K^\infty_{\nu,\tilde k.-\tilde k}}\left(N+\frac{2\pi n\lambda^2}{b^2\sin\pi\tilde k}+\mathrm{const}\right)\right).$$

The operators $\partial^k\varphi\,\bar\partial^k\varphi\,V_\nu$

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For $N=0$ we obtain the vacuum expectation value

$$\langle\mathscr{T}_n(\partial^k\varphi\,\bar\partial^k\varphi\,V_\nu)\rangle = -2n\left(\frac{m}{2}\right)^{2\tilde k}\frac{\Gamma^2(k)\Gamma(1-\tilde k)}{\Gamma(\tilde k)}\frac{K^\infty_{\nu,\tilde k}}{K^\infty_{\nu,-\tilde k}}+O(b^2).$$

This case is more special, because even in the term without interaction $\overline{(\bar\partial^k\varphi)}_{r_0}\mathbf{a}_{-k}|0\rangle_\nu = 2\mathrm{i}\overline{(\bar\partial^k f_{\tilde k})}_{r_0}|0\rangle_\nu \neq 0$.
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It can be shown that at the poles $\tilde k = 2,4,\ldots$ the vacuum expectation values are in fact finite but of the order b^{-2}. They also can be calculated.

Conclusion

- The semiclassical approach allows one to calculate form factors for small b on the background of a classical solution. The classical solution is determined by the exponential operator at the origin.
- The basis for quantum fluctuations about the classical solutions is given by an equation that generalize the Bessel equation ('sinh-Bessel equation').
- In the leading order in b quantum contributions appear for descendant operators. Moreover, the interaction terms in the also contribute the leading order.
- The contributions of the interactions can be effectively found by use of inhomogeneous sinh-Bessel equations.

Beyond this talk:

- The operators with two chiralities demand the renormalization consistent with that of the conformal perturbation theory.
- The operators $\partial\bar{\partial}\varphi\, e^{\alpha\varphi}$ (and, probably, all other mixed derivative operators) vanish after renormalization.
- The mathematical part of the technique is deeply related to the Tracy–Widom approach to the classical sinh-Gordon model. The sinh-Bessel function can be naturally interpreted and treated in this approach. Moreover, they admit a generalization to non-integer subscripts k, which can be used in the description of operators at the branch points of folded plane. (Not discussed in the talk.)